

# A Beale–Kato–Majda criterion for the 3-D Compressible Nematic Liquid Crystal Flows with Vacuum \*

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## Abstract

In this paper, we prove a Beale–Kato–Majda blow-up criterion in terms of the gradient of the velocity only for the strong solution to the 3-D compressible nematic liquid crystal flows with nonnegative initial densities. More precisely, the strong solution exists globally if the  $L^1(0, T; L^\infty)$ -norm of the gradient of the velocity  $u$  is bounded. Our criterion improves the recent result of X. Liu and L. Liu ([25], A blow-up criterion for the compressible liquid crystals system, arXiv:1011.4399v2 [math-ph] 23 Nov. 2010).

**Keywords:** Compressible nematic liquid crystal flows; strong solution; blow-up criterion; Compressible Navier–Stokes equations

**2010 AMS Subject Classification:** 76A15, 76N10, 35B65, 35Q35

## 1 Introduction

The governing system of equations for the compressible nematic liquid (NLC) crystal flows is the following system of scalar or vector fields  $\rho(t, x)$ ,  $u(t, x)$  and  $d(t, x)$  for  $(t, x) \in (0, +\infty) \times \Omega$ , for a bounded smooth domain  $\Omega \subset \mathbb{R}^3$ :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \mu \Delta u - \lambda \nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2}(|\nabla d|^2 + F(d)I)), \\ \partial_t d + (u \cdot \nabla)d = \nu(\Delta d - f(d)) \end{cases} \quad (1.1)$$

together with the initial value conditions:

$$\rho(0, x) = \rho_0(x) \geq 0, \quad u(0, x) = u_0(x), \quad d(0, x) = d_0(x), \quad \forall x \in \Omega, \quad (1.2)$$

and the boundary value conditions:

$$u(t, x) = 0, \quad d(t, x) = d_0(x), \quad |d_0(x)| = 1, \quad \forall (t, x) \in [0, +\infty) \times \partial\Omega. \quad (1.3)$$

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\*Research supported by the National Natural Science Foundation of China (11171357).

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Here we denote by  $\rho, u = (u_1, u_2, u_3), d = (d_1, d_2, d_3)$  the unknown density, velocity and orientation parameter of liquid crystal, respectively, and  $P = P(\rho)$  is the pressure function. Besides,  $\mu, \lambda$  and  $\nu$  are positive viscosity coefficients. The non-standard term  $\nabla d \odot \nabla d$  denotes the  $3 \times 3$  matrix, whose  $(i, j)$ -th element is given by  $\sum_{k=1}^3 \partial_i d_k \partial_j d_k$ .  $I$  is the unit matrix.  $f(d)$  is a polynomial function of  $d$  which satisfies  $f(d) = \frac{\partial}{\partial d} F(d)$ , where  $F(d)$  is the bulk part of the elastic energy; usually we choose  $F(d)$  to be the Ginzburg–Landau penalization, i.e.,  $F(d) = \frac{1}{4\sigma^2}(|d|^2 - 1)^2$  and  $f(d) = \frac{1}{\sigma^2}(|d|^2 - 1)d$ , where  $\sigma$  is a positive constant. In what follows, we will assume  $\sigma = 1$  since its specific value does not play a special role in our discussion. Besides, we assume that the pressure function  $P$  satisfies

$$P = P(\cdot) \in C^1[0, \infty), \quad P(0) = 0. \quad (1.4)$$

The above system (1.1) is a simplified version of Ericksen–Leslie system modeling the flow of compressible nematic liquid crystals, and the hydrodynamic theory of liquid crystals was established by Ericksen [5, 6] and Leslie [17] in the 1960’s. When  $d \equiv 0$ , the system becomes to the compressible Navier–Stokes (CNS) equations. Matsumura and Nishida [27] obtained global existence of smooth solutions for the initial data is a small perturbation of a non-vacuum equilibrium. For the existence of solutions for arbitrary initial value, Lions [18] and Feireisl [9] established the global existence of weak solution to the CNS equations. Cho et al. [2, 3, 4] proved that the existence and uniqueness of local strong solutions of the CNS equations in the case where initial density need not to be positive and may vanish in an open set. Xin in [32] showed that there is no global smooth solution to the Cauchy problem of the CNS equations with a nontrivial compactly supported initial density. Hence, there are many works [3, 7, 8, 12, 13, 14, 30, 31] try to establish blow-up criterion for the strong solution to the CNS equations. In particular, it is proved in [14] by Huang, Li and Xin that the serrin’s blow-up criterion (see [28]) for the incompressible Navier–Stokes equations still holds for the CNS equations, i.e., if  $T^*$  is the maximal time of existence strong solution, then

$$\lim_{T \rightarrow T^*} (\|\operatorname{div} u\|_{L^1(0, T; L^\infty)} + \|\rho^{\frac{1}{2}} u\|_{L^s(0, T; L^r)}) = \infty \quad (1.5)$$

or

$$\lim_{T \rightarrow T^*} (\|\rho\|_{L^1(0, T; L^\infty)} + \|\rho^{\frac{1}{2}} u\|_{L^s(0, T; L^r)}) = \infty, \quad (1.6)$$

where  $r$  and  $s$  satisfy  $\frac{2}{s} + \frac{3}{r} \leq 1$ ,  $3 < r \leq \infty$ . In [12, 13], Huang et al. established that the Beale–Kato–Majda criterion (see [1]) for the ideal incompressible flows still hold for the CNS equations, that is

$$\lim_{T \rightarrow T^*} \int_0^T \|\nabla u\|_{L^\infty} dt = \infty.$$

Sun, Wang and Zhang in [30] (see also [14]) obtained another Beale–Kato–Majda criterion in terms of the density, i.e.,

$$\lim_{T \rightarrow T^*} \sup \|\rho\|_{L^\infty(0, T; L^\infty)} = \infty.$$

When  $\rho$  is a positive constant, the system (1.1) becomes to the incompressible nematic liquid crystal (INLC) equations, the global-in-time weak solutions and local-in-time strong solution have been studied by Lin and Liu [20, 21]. In [11], Hu and Wang established global existence of strong solutions and weak-strong uniqueness for initial data belonging to the Besov spaces of positive order under some smallness assumptions. Liu and Cui in [24] obtained that the blow-up criterion (1.5) or (1.6) still holds for the solution of the INLC equations. We also refer [10, 19, 22, 23, 29] and the reference cited therein for other related work on the INLC equations.

Inspired by the above mentioned works on blow-up criterion of strong solution of CNS and INLC equations, particularly the results of Huang et al. [12, 13] and Sun et al. [30, 31], we want to investigate a similar problem for the compressible nematic liquid crystal flow (1.1)–(1.3). Before stating the main result, we denote the following simplified notations of Sobolev spaces

$$L^q := L^q(\Omega), \quad W^{k,p} := W^{k,p}(\Omega), \quad H^k := H^k(\Omega), \quad H_0^1 := H_0^1(\Omega).$$

When the initial vacuum is allowed, the well-posedness and blow-up criterion for strong solutions to the compressible nematic liquid crystal flows (1.1)–(1.3) have been investigated by Liu et al. in [25, 26]. Here, we write down the main results of Liu et al. [25, 26].

**Theorem 1.1** *Suppose that the initial value  $(\rho_0, u_0, d_0)$  satisfies the following regularity conditions*

$$0 \leq \rho_0 \in W^{1,6}, \quad u_0 \in H_0^1 \cap H^2 \quad \text{and} \quad d_0 \in H^3,$$

*and the compatibility condition*

$$\mu \Delta u_0 - \lambda \operatorname{div}(\nabla d_0 \odot \nabla d_0 - \frac{1}{2}(|\nabla d_0|^2 + F(d_0))) - \nabla P(\rho_0) = \sqrt{\rho} g \text{ for some } g \in L^2. \quad (1.7)$$

*Then there exist a small  $T \in (0, \infty)$  and a unique strong solution  $(\rho, u, d)$  to the system (1.1) with initial boundary condition (1.2)–(1.3) such that*

$$\begin{aligned} 0 \leq \rho &\in C([0, T]; W^{1,6}), & \rho_t &\in C([0, T]; L^6), \\ u &\in C([0, T]; H_0^1 \cap H^2) \cap L^2(0, T; W^{2,6}), & u_t &\in L^2(0, T; H_0^1), \\ d &\in C([0, T]; H^3), & d_t &\in C([0, T]; H_0^1) \cap L^2(0, T; H^2), \\ d_{tt} &\in L^2(0, T; L^2), & \sqrt{\rho} u_t &\in C([0, T]; L^2). \end{aligned}$$

*Moreover, let  $T^*$  be the maximal existence time of the solution. If  $T^* < \infty$ , then there holds*

$$\lim_{T \rightarrow T^*} \int_0^T (\|\nabla u\|_{L^\alpha}^\beta + \|u\|_{W^{1,\infty}}) dt = \infty, \quad (1.8)$$

*where  $\alpha, \beta$  satisfying  $\frac{3}{\alpha} + \frac{2}{\beta} < 2$  and  $\beta \geq 4$ .*

**Remark 1.1** Another similar system of partial differential equations modeling compressible nematic liquid crystal flows has been studied by Huang, Wang and Wen in [15, 16]. They obtained the existence of local in time strong solution and two blow-up criteria under some suitable assumption condition  $u$  and  $d$  or  $\rho$  and  $d$ .

The purpose of this paper is to obtain the Beale–Kato–Majda blow-up criterion only in terms of the gradient of the velocity still holds for the liquid crystal flows. Our main result is the following

**Theorem 1.2** *Assume that  $(\rho, u, d)$  is the strong solution constructed in Theorem 1.1, and  $T^*$  be the maximal existence time of the solution. If  $T^* < \infty$ , then we have*

$$\limsup_{T \rightarrow T^*} \|\nabla u\|_{L^1(0, T; L^\infty)} = \infty. \quad (1.9)$$

The proof of this theorem will be given in the next section. As a standard practice, we will show that if (1.9) does not hold then the strong solution  $(\rho, u, d)$  can be extended beyond the time  $T^*$ . To this end we will step-by-step establish a series of higher-order norm estimates for the strong solution  $(\rho, u, d)$ . The key fact used in this deduction is that the boundedness of the  $L^1(0, T; L^\infty)$ -norm of  $\nabla u$  implies both boundedness of the  $L^\infty(0, T; L^\infty)$ -norm of the density  $\rho$  and boundedness of the  $L^\infty(0, T; W^{1, q})$ -norm of  $d$  with  $2 \leq q \leq \infty$ .

## 2 Proof of Theorem 1.2

Let  $(\rho, u, d)$  be the unique strong solution to the system (1.1) with initial-boundary condition (1.2)–(1.3). We assume that the opposite to (1.9) holds, i.e.,

$$\lim_{T \rightarrow T^*} \|\nabla u\|_{L^1(0, T; L^\infty)} \leq M < \infty. \quad (2.1)$$

In what follows, we note that  $C$  denotes a generic constant depending only on  $\mu, \lambda, \nu, M, T, \Omega$  and the initial data. By using the mass conservation equation (1.1)<sub>1</sub> and the assumption (2.1), it is easy to obtain the  $L^\infty$ -norm bounds of the density,

**Lemma 2.1** *Assume that*

$$\int_0^T \|\operatorname{div} u\|_{L^\infty} dt \leq C, \quad 0 \leq T < T^*, \quad (2.2)$$

*then*

$$\|\rho\|_{L^\infty(0, T; L^\infty)} \leq C \quad \forall 0 \leq T < T^*. \quad (2.3)$$

**Proof.** The proof is essentially due to Huang and Xin [12], for reader's convenience, we sketch it here.

Multiplying the mass conservation equation (1.1)<sub>1</sub> by  $q\rho^{q-1}$  with  $q > 1$ , it follows that

$$\partial_t(\rho^q) + \operatorname{div}(\rho^q u) + (q-1)\rho^q \operatorname{div} u = 0.$$

Integrating the above equality over  $\Omega$  yields

$$\partial_t \|\rho\|_{L^q}^q \leq (q-1) \|\operatorname{div} u\|_{L^\infty} \|\rho\|_{L^q}^q,$$

i.e.,

$$\partial_t \|\rho\|_{L^q} \leq \frac{(q-1)}{q} \|\operatorname{div} u\|_{L^\infty} \|\rho\|_{L^q}. \quad (2.4)$$

The condition (2.2) and the estimate (2.3) imply that

$$\partial_t \|\rho\|_{L^q} \leq C \quad \text{for } \forall q > 1,$$

where  $C$  is a positive constant independent of  $q$ , letting  $q \rightarrow \infty$ , we obtain (2.3), and this completes the proof of the lemma.  $\square$

According to the assumption (1.4) on the pressure  $P$  and Lemma 2.1, it is easy to obtain

$$\sup_{0 \leq t \leq T} \{\|P(\rho)\|_{L^\infty}, \|P'(\rho)\|_{L^\infty}\} \leq C < \infty. \quad (2.5)$$

Now, let us derive the stand energy inequality.

**Lemma 2.2** *There holds*

$$\sup_{0 \leq t \leq T} \int_{\Omega} (\rho|u|^2 + |\nabla d|^2 + 2F(d)) dx + \int_0^T \int_{\Omega} |\nabla u|^2 dx dt + \int_0^T \int_{\Omega} |\Delta d - f(d)|^2 dx dt \leq C. \quad (2.6)$$

**Proof.** Multiplying the momentum equation (1.1)<sub>2</sub> by  $u$ , integrating over  $\Omega$  and making use of the mass conversation equation (1.1)<sub>1</sub>, it follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho|u|^2 dx + \mu \int_{\Omega} |\nabla u|^2 dx = - \int_{\Omega} u \nabla P dx - \lambda \int_{\Omega} (u \cdot \nabla) d \cdot (\Delta d - f(d)) dx, \quad (2.7)$$

where we have used the fact that  $\operatorname{div}(\nabla d \odot \nabla d) = (\nabla d)^T \Delta d - \nabla \cdot \frac{|\nabla d|^2}{2}$ . Multiplying the liquid crystal equation (1.1)<sub>3</sub> by  $\Delta d - f(d)$  and integrating over  $\Omega$ , we obtain

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} |\nabla d|^2 + F(d) \right) dx + \nu \int_{\Omega} |\Delta d - f(d)|^2 dx = \int_{\Omega} (u \cdot \nabla) d \cdot (\Delta d - f(d)) dx. \quad (2.8)$$

Combining (2.7) and (2.8) together

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} (\rho|u|^2 + \lambda |\nabla d|^2) + \lambda F(d) \right] dx + \mu \int_{\Omega} |\nabla u|^2 dx + \lambda \nu \int_{\Omega} |\Delta d - f(d)|^2 dx \\ &= - \int_{\Omega} u \nabla P dx = \int_{\Omega} P \operatorname{div} u dx \leq \varepsilon \int_{\Omega} |\nabla u|^2 dx + C\varepsilon^{-1}, \end{aligned} \quad (2.9)$$

where we have used the estimates (2.3), (2.5) and the Young inequality. Taking  $\varepsilon$  small enough and applying the Gronwall's inequality, we can establish the estimate (2.6) immediately.  $\square$

In the next lemma, we will derive some estimates of  $d$ .

**Lemma 2.3** *Under the assumption (2.1), it holds that for  $0 \leq T < T^*$*

$$\sup_{0 \leq t \leq T} (\|d\|_{L^q} + \|\nabla d\|_{L^q}) \leq C \quad \text{for all } 2 \leq q \leq \infty; \quad (2.10)$$

$$\sup_{0 \leq t \leq T} \|\nabla d\|_{L^2}^2 + \int_0^T \|d_t\|_{L^2}^2 dt \leq C; \quad (2.11)$$

**Proof.** We first multiplying the liquid crystal equation (1.1)<sub>3</sub> by  $q|d|^{q-2}d$  with  $q \geq 2$ , and integrating over  $\Omega$ , then there holds

$$\begin{aligned}
& \frac{d}{dt} \|d\|_{L^q}^q + \int_{\Omega} (q\nu |\nabla d|^2 |d|^2 + q(q-2)\nu |d|^{q-2} |\nabla |d||^2) dx \\
&= - \sum_{i=1}^3 \int_{\Omega} u_i \partial_i (|d|^q) dx - q\nu \int_{\Omega} |d|^{q+2} dx + q\nu \int_{\Omega} |d|^q dx \\
&= - \sum_{i=1}^3 \int_{\Omega} \partial_i u_i |d|^q dx - q\nu \int_{\Omega} |d|^{q+2} dx + q\nu \int_{\Omega} |d|^q dx \\
&\leq C(\|\nabla u\|_{L^\infty} + 1) \|d\|_{L^q}^q.
\end{aligned}$$

By using the Gronwall's inequality, one obtains the inequality

$$\sup_{0 \leq t \leq T} \|d\|_{L^q} \leq C \quad \text{for all } q \geq 2. \quad (2.12)$$

By letting  $q \rightarrow \infty$ , we notice that the estimate (2.12) still holds.

Multiplying the gradient of the liquid crystal equation (1.1)<sub>3</sub> by  $q|\nabla d|^{q-2}\nabla d$  with  $q \geq 2$ , and integrating over  $\Omega$ , then there holds

$$\begin{aligned}
& \frac{d}{dt} \|\nabla d\|_{L^q}^q + \int_{\Omega} (q\nu |\nabla(\nabla d)|^2 |\nabla d|^{q-2} + q(q-2)\nu |\nabla |\nabla d||^2 |\nabla d|^{q-2}) dx \\
&= - \sum_{i=1}^3 \int_{\Omega} u_i \partial_i (|\nabla d|^q) dx - \sum_{i=1}^3 q \int_{\Omega} \nabla u_i \partial_i d |\nabla d|^{q-2} \nabla d dx - \nu q \int_{\Omega} \nabla(|d|^2 d) |\nabla d|^{q-2} \nabla d dx + \nu q \int_{\Omega} |\nabla d|^q dx \\
&= - \sum_{i=1}^3 \int_{\Omega} \partial_i u_i |\nabla d|^q dx - \sum_{i=1}^3 q \int_{\Omega} \nabla u_i \partial_i d |\nabla d|^{q-2} \nabla d dx - \nu q \int_{\Omega} \nabla(|d|^2 d) |\nabla d|^{q-2} \nabla d dx + \nu q \int_{\Omega} |\nabla d|^q dx \\
&\leq C \|\nabla u\|_{L^\infty} \|\nabla d\|_{L^q}^q + \nu q \|\nabla d\|_{L^q}^q - \nu q \int_{\Omega} \nabla(|d|^2 d) |\nabla d|^{q-2} \nabla d dx \\
&= (C \|\nabla u\|_{L^\infty} + \nu q) \|\nabla d\|_{L^q}^q - \nu q \int_{\Omega} |d|^2 \nabla d |\nabla d|^{q-2} \nabla d dx - \nu q \int_{\Omega} d \nabla(|d|^2) |\nabla d|^{q-2} \nabla d dx \\
&= (C \|\nabla u\|_{L^\infty} + \nu q) \|\nabla d\|_{L^q}^q - 3\nu q \int_{\Omega} |d|^2 |\nabla d|^q dx \\
&\leq C(\|\nabla u\|_{L^\infty} + 1) \|\nabla d\|_{L^q}^q,
\end{aligned}$$

where we have used the fact that  $\nabla |d|^2 = 2|d|\nabla |d| = 2|d|\frac{d \cdot \nabla d}{|d|} = 2d \nabla d$  in the last equality. By using the Gronwall's inequality again, we obtain

$$\sup_{0 \leq t \leq T} \|\nabla d\|_{L^q} \leq C \quad \text{for all } q \geq 2. \quad (2.13)$$

Letting  $q \rightarrow \infty$ , estimate (2.13) still holds, and the inequalities (2.12) and (2.13) imply that estimate (2.10) holds.

To prove the estimate (2.11), we multiplying the liquid crystal equation (1.1)<sub>3</sub> by  $d_t$  and integrating over  $\Omega$ , then

$$\|d_t\|_{L^2}^2 + \frac{\nu}{2} \frac{d}{dt} \int_{\Omega} |\nabla d|^2 dx = - \int_{\Omega} (u \cdot \nabla) d d_t dx - \nu \int_{\Omega} f(d) d_t dx$$

$$\begin{aligned}
&\leq C(\|u\|_{L^2}\|\nabla d\|_{L^\infty}\|d_t\|_{L^2} + \|d\|_{L^\infty}^2\|d\|_{L^2}\|d_t\|_{L^2} + \|d\|_{L^2}\|d_t\|_{L^2}) \\
&\leq \frac{1}{2}\|d_t\|_{L^2}^2 + C,
\end{aligned}$$

where we have used the estimates (2.6) and (2.10). Integrating the above inequality over  $[0, T]$  gives the estimate (2.11).  $\square$

For function  $f \in \Omega \times (0, T)$ , let

$$\dot{f} = f_t + u \cdot \nabla f$$

denote the material derivative of the function  $f$ . Then we have following lemma.

**Lemma 2.4** *Under the assumption (2.1), it holds that for  $0 \leq T < T^*$*

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|d\|_{H^2}^2) + \int_0^T \int_\Omega (\rho|\dot{u}|^2 + |\nabla d_t|^2) dx dt \leq C; \quad (2.14)$$

$$\int_0^T \|\nabla d\|_{H^2}^2 dt \leq C. \quad (2.15)$$

**Proof.** Noticing that the momentum equation (1.1)<sub>2</sub> can be rewrote as

$$\rho \dot{u} + \nabla P = \mu \Delta u - \lambda (\nabla d)^T (\Delta d - f(d)). \quad (2.16)$$

Multiplying the equation (2.16) above by  $\dot{u}$  and integrating over  $\Omega$ , one obtains the equality

$$\begin{aligned}
&\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \int_\Omega \rho |\dot{u}|^2 dx \\
&= \mu \int_\Omega u \cdot \nabla u \Delta u dx + \int_\Omega P \operatorname{div} u_t dx - \int_\Omega u \cdot \nabla u \nabla P dx \\
&\quad - \lambda \int_\Omega (u_t \cdot \nabla) d (\Delta d - f(d)) dx - \lambda \int_\Omega (u \cdot \nabla) u \cdot \nabla d (\Delta d - f(d)) dx
\end{aligned} \quad (2.17)$$

Combining the mass conservation equation (1.1)<sub>1</sub> and the assumption (2.1), it follows that the pressure  $P$  satisfies the following equation

$$P_t + P'(\rho) \nabla \rho \cdot u + P'(\rho) \rho \operatorname{div} u = 0. \quad (2.18)$$

Hence, we have

$$\begin{aligned}
\int_\Omega P \operatorname{div} u_t dx &= \frac{d}{dt} \int_\Omega P \operatorname{div} u dx - \int_\Omega P_t \operatorname{div} u dx \\
&= \frac{d}{dt} \int_\Omega P \operatorname{div} u dx + \int_\Omega P'(\rho) (\nabla \rho \cdot u + \rho \operatorname{div} u) \operatorname{div} u dx.
\end{aligned} \quad (2.19)$$

To estimate the term  $-\lambda \int_\Omega (u_t \cdot \nabla) d (\Delta d - f(d)) dx$ , we have

$$-\lambda \int_\Omega (u_t \cdot \nabla) d (\Delta d - f(d)) dx = \lambda \sum_{i,j=1}^3 \left( \int_\Omega \partial_j u_{it} \partial_i d \partial_j d dx + \int_\Omega u_{it} \partial_i \partial_j d \partial_j d dx \right) + \lambda \int_\Omega u_t \cdot \nabla df(d) dx$$

$$\begin{aligned}
&= \lambda \sum_{i,j=1}^3 \left( \int_{\Omega} \partial_j u_{it} \partial_i d \partial_j d \, dx - \frac{1}{2} \int_{\Omega} \partial_i u_{it} |\partial_j d|^2 \, dx \right) - \lambda \sum_{i=1}^3 \int_{\Omega} \partial_i u_{it} \left( \frac{|d|^4}{4} - \frac{|d|^2}{2} \right) \, dx \\
&= \lambda \sum_{i,j=1}^3 \left\{ \frac{d}{dt} \int_{\Omega} (\partial_j u_i \partial_i d \partial_j d - \frac{1}{2} \partial_i u_i |\partial_j d|^2) \, dx - \int_{\Omega} \partial_j u_i \partial_i d_t \partial_j d \, dx - \int_{\Omega} \partial_j u_i \partial_i d \partial_j d_t \, dx \right. \\
&\quad \left. + \int_{\Omega} \partial_i u_i \partial_j d_t \partial_j d \, dx - \frac{d}{dt} \int_{\Omega} \left( \frac{\partial_i u_i |d|^4}{4} - \frac{\partial_i u_i |d|^2}{2} \right) \, dx + \int_{\Omega} \partial_i u_i (|d|^3 d_t - |d| d_t) \, dx \right\} \\
&\leq \lambda \sum_{i,j=1}^3 \frac{d}{dt} \int_{\Omega} (\partial_j u_i \partial_i d \partial_j d - \frac{1}{2} \partial_i u_i |\partial_j d|^2 - \frac{1}{4} \partial_i u_i |d|^4 + \frac{1}{2} \partial_i u_i |d|^2) \, dx \\
&\quad + C \|\nabla u\|_{L^2} \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} + C \|\nabla u\|_{L^2} \|d_t\|_{L^2} (\|d\|_{L^\infty}^2 + 1) \|d\|_{L^\infty} \\
&\leq \lambda \sum_{i,j=1}^3 \frac{d}{dt} \int_{\Omega} (\partial_j u_i \partial_i d \partial_j d - \frac{1}{2} \partial_i u_i |\partial_j d|^2 - \frac{1}{4} \partial_i u_i |d|^4 + \frac{1}{2} \partial_i u_i |d|^2) \, dx \\
&\quad + C \varepsilon^{-1} \|\nabla u\|_{L^2}^2 + \varepsilon \|\nabla d_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C \|d_t\|_{L^2}^2, \tag{2.20}
\end{aligned}$$

where we have used estimate (2.10) in the last inequality. Inserting (2.19) and (2.20) into (2.17), and integrating over  $[0, T]$  give that

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2 + \int_0^T \int_{\Omega} \rho |\dot{u}|^2 \, dx \, dt \\
&\leq C + C \int_0^T \int_{\Omega} u \cdot \nabla u \Delta u \, dx \, dt + C \int_{\Omega} P(\rho) \operatorname{div} u \, dx(T) + C \int_0^T \int_{\Omega} P'(\rho) (\nabla \rho \cdot u + \rho \operatorname{div} u) \operatorname{div} u \, dx \, dt \\
&\quad + C \sum_{i,j=1}^3 \int_{\Omega} (\partial_j u_i \partial_i d \partial_j d - \frac{1}{2} \partial_i u_i |\partial_j d|^2 - \frac{1}{4} \partial_i u_i |d|^4 + \frac{1}{2} \partial_i u_i |d|^2) \, dx(T) \\
&\quad + \varepsilon \int_0^T \|\nabla d_t\|_{L^2}^2 \, dt + C \varepsilon^{-1} \int_0^T \|\nabla u\|_{L^2}^2 \, dt + C \int_0^T (\|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2) \, dt \\
&\quad + \int_0^T \int_{\Omega} |u| |\nabla u| |\nabla P| \, dx \, dt + C \int_0^T \int_{\Omega} |u| |\nabla u| |\nabla d| (|\Delta d| + |f(d)|) \, dx \, dt \\
&\leq C + \varepsilon \int_0^T \|\nabla d_t\|_{L^2}^2 \, dt + C \int_0^T \int_{\Omega} u \cdot \nabla u \Delta u \, dx \, dt + C \int_{\Omega} P(\rho) \operatorname{div} u \, dx(T) \\
&\quad + C \int_0^T \int_{\Omega} P'(\rho) (\nabla \rho \cdot u + \rho \operatorname{div} u) \operatorname{div} u \, dx \, dt \\
&\quad + C \sum_{i,j=1}^3 \int_{\Omega} (\partial_j u_i \partial_i d \partial_j d - \frac{1}{2} \partial_i u_i |\partial_j d|^2 - \frac{1}{4} \partial_i u_i |d|^4 + \frac{1}{2} \partial_i u_i |d|^2) \, dx(T) \\
&\quad + \int_0^T \int_{\Omega} |u| |\nabla u| |\nabla P| \, dx \, dt + C \int_0^T \int_{\Omega} |u| |\nabla u| |\nabla d| (|\Delta d| + |f(d)|) \, dx \, dt, \tag{2.21}
\end{aligned}$$

where we have used the estimate (2.6) and (2.11). To estimate the terms on the right side of (2.21), by using Lemma 2.1, the estimates (2.6), (2.10) and (2.11), we get

$$\int_0^T \int_{\Omega} u \cdot \nabla u \Delta u \, dx \, dt = \sum_{i,j=1}^3 \int_0^T \int_{\Omega} (-\partial_j u_i \partial_i u \partial_j u - u_i \partial_i \partial_j u \partial_j u) \, dx \, dt$$



$$\begin{aligned}
&= \sum_{i,j=1}^3 \int_0^T \int_{\Omega} (-\partial_j u_i \partial_i u \partial_j u + \frac{1}{2} \partial_i u_i |\partial_j u|^2) dx dt \\
&\leq C \int_0^T \|\nabla u\|_{L^3}^3 dt \leq C \int_0^T \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2 dt
\end{aligned} \tag{2.22}$$

$$\int_{\Omega} P(\rho) \operatorname{div} u dx(T) \leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C \int_{\Omega} |P(\rho)|^2 dx \leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C; \tag{2.23}$$

$$\begin{aligned}
\int_0^T \int_{\Omega} P'(\rho) (\nabla \rho \cdot u) \operatorname{div} u dx dt &\leq C \int_0^T \|\nabla \rho\|_{L^2} \|u\|_{L^2} \|\operatorname{div} u\|_{L^\infty} dt \\
&\leq C \int_0^T \|\nabla \rho\|_{L^2}^2 \|\nabla u\|_{L^\infty} dt + C \int_0^T \|u\|_{L^2}^2 \|\nabla u\|_{L^\infty} dt \\
&\leq C \int_0^T \|\nabla \rho\|_{L^2}^2 \|\nabla u\|_{L^\infty} dt + C \int_0^T \|\nabla u\|_{L^\infty} dt \\
&\leq C \int_0^T \|\nabla \rho\|_{L^2}^2 \|\nabla u\|_{L^\infty} dt + C;
\end{aligned} \tag{2.24}$$

$$\int_0^T \int_{\Omega} P'(\rho) \rho |\operatorname{div} u|^2 dx dt \leq C \int_0^T \|\rho\|_{L^\infty} \|\nabla u\|_{L^2}^2 dt \leq C \int_0^T \|\nabla u\|_{L^2}^2 dt \leq C; \tag{2.25}$$

$$\begin{aligned}
&\sum_{i,j=1}^3 \int_{\Omega} (\partial_j u_i \partial_i d \partial_j d - \frac{1}{2} \partial_i u_i |\partial_j d|^2 - \frac{1}{4} \partial_i u_i |d|^4 + \frac{1}{2} \partial_i u_i |d|^2) dx(T) \\
&\leq C (\|\nabla u\|_{L^2} \|\nabla d\|_{L^2} \|\nabla d\|_{L^\infty} + \|\nabla u\|_{L^2} \|d\|_{L^2} (\|d\|_{L^\infty}^3 + \|d\|_{L^\infty})) \\
&\leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C \|\nabla d\|_{L^2}^2 \|\nabla d\|_{L^\infty}^2 + C \|d\|_{L^2}^2 (\|d\|_{L^\infty}^6 + \|d\|_{L^\infty}^2) \\
&\leq \frac{1}{4} \|\nabla u\|_{L^2}^2 + C
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
\int_0^T \int_{\Omega} |u| |\nabla u| |\nabla P| dx dt &\leq C \int_0^T \int_{\Omega} |u| |\nabla u| |\nabla \rho| dx dt \\
&\leq C \int_0^T \|u\|_{L^6} \|\nabla u\|_{L^3} \|\nabla \rho\|_{L^2} dt \leq C \int_0^T \|\nabla u\|_{L^2}^{\frac{5}{3}} \|\nabla u\|_{L^\infty}^{\frac{1}{3}} \|\nabla \rho\|_{L^2} dt \\
&\leq C \int_0^T (\|\nabla u\|_{L^2}^2 \|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^{\frac{2}{3}} \|\nabla u\|_{L^2}^{\frac{4}{3}}) dt \\
&\leq C \int_0^T (\|\nabla u\|_{L^2}^2 \|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2 + C) dt;
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
\int_0^T \int_{\Omega} |u| |\nabla u| |\nabla d| (|\Delta d| + |f(d)|) dx dt &\leq C \int_0^T (\|u\|_{L^6} \|\nabla u\|_{L^6} \|\nabla d\|_{L^6} \|\Delta d\|_{L^2} \\
&\quad + \|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla d\|_{L^3} (\|d\|_{L^\infty}^3 + \|d\|_{L^\infty})) dt \\
&\leq C \int_0^T (\|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\Delta d\|_{L^2} + \|\nabla u\|_{L^2}^2) dt \\
&\leq \int_0^T (\varepsilon \|\nabla^2 u\|_{L^2}^2 + C \varepsilon^{-1} \|\nabla u\|_{L^2}^2 \|\Delta d\|_{L^2}^2) dt + C.
\end{aligned} \tag{2.28}$$

By using the stand elliptic regularity result to (2.16), we have

$$\|\nabla^2 u\|_{L^2}^2 \leq \|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C (\|\nabla u\|_{L^2}^2 + \|\rho \dot{u}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|(\nabla d)^T (\Delta d - f(d))\|_{L^2}^2)$$

$$\begin{aligned}
&\leq C(\|\nabla u\|_{L^2}^2 + \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2 (\|\Delta d\|_{L^2}^2 + \|f(d)\|_{L^2}^2)) \\
&\leq C(\|\nabla u\|_{L^2}^2 + \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + 1).
\end{aligned} \tag{2.29}$$

Combining estimates (2.21)–(2.29) and taking  $\varepsilon$  small enough, we can get

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2 + \int_0^T \int_\Omega \rho |\dot{u}|^2 dx dt \\
&\leq C + \varepsilon \int_0^T \|\nabla d_t\|_{L^2}^2 dt + C \int_0^T (\|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) (\|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^2}^2 + 1) dt.
\end{aligned} \tag{2.30}$$

To estimate the orientation parameter  $d$ , by the standard elliptic regularity result to the liquid crystal equation (1.1)<sub>3</sub>, one obtains that

$$\begin{aligned}
\|\nabla^3 d\|_{L^2} &\leq C(\|\nabla d_t\|_{L^2} + \|\nabla(u \cdot \nabla d)\|_{L^2} + \|\nabla f(d)\|_{L^2} + \|d_0\|_{H^3}) \\
&\leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla d\|_{L^\infty} + \|u\| \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^2} (\|d\|_{L^\infty}^2 + \|d\|_{L^\infty}) + \|d_0\|_{H^3}) \\
&\leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \|u\| \|\nabla^2 d\|_{L^2} + C)
\end{aligned} \tag{2.31}$$

Multiplying the liquid crystal equation (1.1)<sub>3</sub> by  $\Delta d_t$ , and integrating over  $\Omega$ , then we have

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \nu |\Delta d|^2 dx + \int_\Omega |\nabla d_t|^2 dx \\
&= \int_\Omega u \cdot \nabla d \Delta d_t dx + \nu \int_\Omega (|d|^2 - 1) d \Delta d_t dx \\
&= \sum_{i,j=1}^3 \int_\Omega u_i \partial_i d \partial_j^2 d_t dx - \nu \int_\Omega \nabla(|d|^2 d) \nabla d_t dx + \nu \int_\Omega \nabla d \nabla d_t dx \\
&= - \sum_{i,j=1}^3 \int_\Omega \partial_j u_i \partial_i d \partial_j d_t dx - \sum_{i,j=1}^3 \int_\Omega u_i \partial_i \partial_j d \partial_j d_t dx - \nu \int_\Omega \nabla(|d|^2 d) \nabla d_t dx + \nu \int_\Omega \nabla d \nabla d_t dx \\
&\leq C(\|\nabla u\|_{L^2} \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} \\
&\quad + \|u \nabla^2 d\|_{L^2} \|\nabla d_t\|_{L^2} + \|d\|_{L^\infty}^2 \|\nabla d\|_{L^2} \|\nabla d_t\|_{L^2} + \|\nabla d\|_{L^2} \|\nabla d_t\|_{L^2}) \\
&\leq \varepsilon \|\nabla d_t\|_{L^2}^2 + C\varepsilon^{-1} (\|\nabla u\|_{L^2}^2 + \int_\Omega |u|^2 |\nabla^2 d|^2 dx + 1),
\end{aligned} \tag{2.32}$$

where we have used the Hölder inequality and estimates (2.10). For the term  $\int_\Omega |u|^2 |\nabla^2 d|^2 dx$ , applying estimate (2.31), we have for  $\eta > 0$

$$\begin{aligned}
\int_\Omega |u|^2 |\nabla^2 d|^2 dx &\leq C \|u\|_{L^6}^2 \|\nabla^2 d\|_{L^3}^2 \leq C \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^6} \|\nabla^3 d\|_{L^2} \\
&\leq \eta \|\nabla^3 d\|_{L^2}^2 + C\eta^{-1} \|\nabla u\|_{L^2}^4 \\
&\leq \eta \|\nabla d_t\|_{L^2}^2 + \eta \int_\Omega |u|^2 |\nabla^2 d|^2 dx + \eta \|\nabla u\|_{L^2}^2 + C\eta^{-1} (\|\nabla u\|_{L^2}^4 + C).
\end{aligned}$$

Hence, taking  $\eta$  small enough

$$\int_\Omega |u|^2 |\nabla^2 d|^2 dx \leq 2\eta \|\nabla d_t\|_{L^2}^2 + C\eta^{-1} (\|\nabla u\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + 1) + C). \tag{2.33}$$

Inserting (2.33) into (2.32), taking  $\varepsilon, \eta$  small enough and integrating above inequality over  $(0; T]$  ensure that

$$\|\Delta d\|_{L^2}^2 + \int_0^T \int_{\Omega} |\nabla d_t|^2 dx dt \leq C \int_0^T \|\nabla u\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + 1) dt + C. \quad (2.34)$$

Now, we will estimate the density  $\rho$ . Applying the operator  $\nabla$  to the mass conservation equation (1.1)<sub>1</sub>, then multiplying it by  $\nabla \rho$  and integrating over  $\Omega$  yield

$$\begin{aligned} \frac{d}{dt} \|\nabla \rho\|_{L^2}^2 &= - \int_{\Omega} |\nabla \rho|^2 \operatorname{div} u dx - 2 \int_{\Omega} \rho \nabla \rho \nabla \operatorname{div} u dx - 2 \int_{\Omega} (\nabla \rho \cdot \nabla u) \nabla \rho dx \\ &\leq C \|\nabla \rho\|_{L^2}^2 \|\nabla u\|_{L^\infty} + C \|\nabla \rho\|_{L^2} \|\nabla \operatorname{div} u\|_{L^2} \\ &\leq \varepsilon \|\nabla^2 u\|_{L^2}^2 + C \varepsilon^{-1} \|\nabla \rho\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + 1) \\ &\leq \varepsilon (\|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + 1) + C \varepsilon^{-1} \|\nabla \rho\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + 1) \\ &\leq \varepsilon \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + C \varepsilon^{-1} \|\nabla \rho\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + 1) + C (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2), \end{aligned}$$

where we have used the estimate (2.29) in the above inequality. Integrating the above estimate over  $(0, T]$  gives that

$$\|\nabla \rho\|_{L^2}^2 \leq \varepsilon \int_0^T \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 dt + \int_0^T (C \varepsilon^{-1} \|\nabla \rho\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + 1) + C (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)) dt \quad (2.35)$$

Combining estimates (2.30), (2.34) and (2.35), and taking  $\varepsilon$  small enough, one obtains that

$$\begin{aligned} \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \int_0^T \int_{\Omega} (\rho |\dot{u}|^2 + |\nabla d_t|^2) dx dt \\ \leq C + C \int_0^T (\|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) (\|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^2}^2 + 1) dt. \end{aligned} \quad (2.36)$$

Since the energy estimate (2.6) implies that  $\int_0^T \|\nabla u\|_{L^2}^2 dt \leq C$ . By using the Gronwall's inequality, the elliptic regularity result  $\|\nabla^2 d\|_{L^2} \leq C (\|\Delta d\|_{L^2} + \|d_0\|_{H^2})$  and noticing that the assumption (2.1), we deduce that the inequality (2.14) holds.

To prove the estimate (2.15), by using the standard elliptic regularity result on (1.1)<sub>3</sub>, we have

$$\begin{aligned} \|\nabla^2 d\|_{L^3}^2 &\leq C (\|d_t\|_{L^3}^2 + \|u \cdot \nabla d\|_{L^3}^2 + \|f(d)\|_{L^3}^2 + \|d_0\|_{H^3}^2) \\ &\leq C (\|d_t\|_{L^2} \|\nabla d_t\|_{L^2} + \|u\|_{L^6}^2 \|\nabla d\|_{L^6}^2 + \|d\|_{L^3}^2 (\|d\|_{L^\infty}^4 + \|d\|_{L^2}^2) + C) \\ &\leq C (\|d_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + C) \\ &\leq C (\|d_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + C), \end{aligned} \quad (2.37)$$

where we have used the estimate (2.14) in the last inequality. Then by using the estimates (2.10), (2.11), (2.14) and the above inequality, we have

$$\begin{aligned} \int_0^T \|\nabla d\|_{H^2}^2 dt &\leq \int_0^T (\|\nabla^3 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) dt \\ &\leq C \int_0^T (\|\nabla d_t\|_{L^2}^2 + \|\nabla(u \cdot \nabla d)\|_{L^2}^2 + \|\nabla f(d)\|_{L^2}^2 + C) dt \end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^T (\|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^\infty}^2 + \|u\|_{L^6}^2 \|\nabla^2 d\|_{L^3}^2 + \|\nabla d\|_{L^2}^2 (\|d\|_{L^\infty}^4 + \|d\|_{L^\infty}^2) + C) dt \\
&\leq C \int_0^T (\|\nabla d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + C) dt \\
&\leq C \int_0^T (\|\nabla d_t\|_{L^2}^2 + \|d_t\|_{L^2}^2 + C) dt \leq C.
\end{aligned}$$

This completes the proof of Lemma 2.4.  $\square$

**Lemma 2.5** *Under the assumption (2.1), it holds that for  $0 \leq T < T^*$*

$$\sup_{0 \leq t \leq T} (\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T (\|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2) dt \leq C. \quad (2.38)$$

**Proof.** Differentiating the momentum equation (1.1)<sub>2</sub> with respect to time, multiplying the resulting equation by  $u_t$ , integrating it over  $\Omega$  and making use of the mass conservation equation (1.1)<sub>1</sub>, one obtains that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 dx + \mu \int_{\Omega} |\nabla u_t|^2 dx - \int_{\Omega} P_t \operatorname{div} u_t dx \\
&= - \int_{\Omega} \rho u \cdot \nabla \left( \frac{|u_t|^2}{2} \right) + (u \cdot \nabla) u u_t + \rho (u_t \cdot \nabla) u u_t dx - \lambda \int_{\Omega} (u_t \cdot \nabla) d_t (\Delta d - f(d)) dt \\
&\quad - \lambda \int_{\Omega} (u_t \cdot \nabla) d (\Delta d - f(d))_t dx \\
&= \int_{\Omega} \nabla \rho \cdot u \frac{|u_t|^2}{2} + \rho \operatorname{div} u \frac{|u_t|^2}{2} - \rho u \cdot \nabla ((u \cdot \nabla) u u_t) - \rho (u_t \cdot \nabla) u u_t dx \\
&\quad - \lambda \int_{\Omega} (u_t \cdot \nabla) d_t (\Delta d - f(d)) dt - \lambda \int_{\Omega} (u_t \cdot \nabla) d (\Delta d - f(d))_t dx. \quad (2.39)
\end{aligned}$$

Differentiating the liquid crystal equation (1.1)<sub>3</sub> with respect to time gives

$$(u_t \cdot \nabla) d = \nu (\Delta d - f(d))_t - d_{tt} - (u \cdot \nabla) d_t.$$

Multiplying the above equality with  $(\Delta d - f(d))_t$  and integrating over  $\Omega$ , one obtains the equality

$$\begin{aligned}
&\int_{\Omega} (u_t \cdot \nabla) d (\Delta d - f(d))_t dx \\
&= \int_{\Omega} (\nu |(\Delta d - f(d))_t|^2 - d_{tt} \Delta d_t + d_{tt} f(d)_t - (u \cdot \nabla) d_t (\Delta d - f(d))_t) dx \\
&= \int_{\Omega} \nu |(\Delta d - f(d))_t|^2 dx + \frac{1}{2} \frac{d}{dt} \|\nabla d_t\|_{L^2}^2 - \int_{\Omega} ((u_t \cdot \nabla) d) f(d)_t dx \\
&\quad + \int_{\Omega} \nu f(d)_t (\Delta d - f(d))_t dx - \int_{\Omega} ((u \cdot \nabla) d_t) \Delta d_t dx \\
&= \int_{\Omega} \nu |(\Delta d - f(d))_t|^2 dx + \frac{1}{2} \frac{d}{dt} \|\nabla d_t\|_{L^2}^2 - \int_{\Omega} ((u_t \cdot \nabla) d) f(d)_t dx \\
&\quad + \int_{\Omega} \nu f(d)_t (\Delta d - f(d))_t dx + \int_{\Omega} ((\nabla u \cdot \nabla) d_t \nabla d_t - \frac{1}{2} \operatorname{div} u |\nabla d_t|^2) dx, \quad (2.40)
\end{aligned}$$

where we have used the fact

$$\begin{aligned}
-\int_{\Omega} ((u \cdot \nabla) d_t) \Delta d_t dx &= -\sum_{i,j=1}^3 \int_{\Omega} u_i \partial_i d_t \partial_{jj} d_t dx \\
&= \sum_{i,j=1}^3 \int_{\Omega} (\partial_j u_i \partial_i d_t \partial_j d_t + u_i \partial_i (\frac{|\partial_j d_t|^2}{2})) dx \\
&= \sum_{i,j=1}^3 (\int_{\Omega} (\partial_j u_i \partial_i d_t \partial_j d_t dx - \frac{1}{2} \int_{\Omega} \partial_i u_i |\partial_j d_t|^2 dx) \\
&= \int_{\Omega} ((\nabla u \cdot \nabla) d_t \nabla d_t - \frac{1}{2} \operatorname{div} u |\nabla d_t|^2) dx
\end{aligned}$$

in the last equality.

From the equation (2.18), we can derive

$$\int_{\Omega} P_t \operatorname{div} u_t dx = - \int_{\Omega} P'(\rho) (\nabla \rho \cdot u + \rho \operatorname{div} u) \operatorname{div} u_t dx. \quad (2.41)$$

Inserting the equalities (2.40) and (2.41) into (2.39) derives

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega} (\frac{1}{2} \rho |u_t|^2 + \lambda |\nabla d_t|^2) dx + \mu \|\nabla u_t\|_{L^2}^2 + \lambda \nu \|(\Delta d - f(d))_t\|_{L^2}^2 \\
&\leq C \int_{\Omega} (|\nabla \rho| |u| |u_t|^2 + \rho |\operatorname{div} u| |u_t|^2 + \rho |u| |u_t| |\nabla u|^2 + \rho |u|^2 |u_t| |\nabla^2 u| + \rho |u|^2 |\nabla u| |\nabla u_t| + \rho |u_t|^2 |\nabla u|) dx \\
&\quad + C \int_{\Omega} (|(u_t \cdot \nabla) df(d)_t| + |(\Delta d - f(d))_t f(d)_t| + |(\nabla u \cdot \nabla) d_t \nabla d_t| + |\operatorname{div} u| |\nabla d_t|^2) dx \\
&\quad + C \int_{\Omega} |(u_t \cdot \nabla) d_t (\Delta d - f(d))| dx + C \int_{\Omega} |p'(\rho)| |\nabla \rho| |u| |\operatorname{div} u_t| + \rho |P'(\rho)| |\operatorname{div} u| |\operatorname{div} u_t| dx \\
&= \sum_{j=1}^{13} J_j. \quad (2.42)
\end{aligned}$$

We will estimate  $J_j$  term by term. In the following calculations, we will make extensive use of Sobolev embedding, Hölder inequality, Lemmas 2.1–2.4 and the estimate (2.5),

$$\begin{aligned}
J_1 &\leq C \|\nabla \rho\|_{L^2} \|u_t\|_{L^6}^2 \|u\|_{L^6} \leq C \|\nabla u_t\|_{L^2}^2 \|\nabla u\|_{L^2} \leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C \varepsilon^{-1}; \\
J_2 + J_6 &\leq C \|\nabla u\|_{L^\infty} \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2; \\
J_7 &\leq C \|u_t\|_{L^2} \|\nabla d\|_{L^3} (\|d\|_{L^\infty}^2 + 1) \|d_t\|_{L^6} \\
&\leq C \|\rho^{\frac{1}{2}} u_t\|_{L^2} \|\nabla d_t\|_{L^2} \leq \varepsilon \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + C \varepsilon^{-1} \|\nabla d_t\|_{L^2}^2; \\
J_8 &\leq C \|(\Delta d - f(d))_t\|_{L^2} \|f(d)_t\|_{L^2} \leq \varepsilon \|(\Delta d - f(d))_t\|_{L^2}^2 + C \varepsilon^{-1} \|d_t\|_{L^2}^2; \\
J_9 + J_{10} &= \int_{\Omega} |(\nabla u \cdot \nabla) d_t \nabla d_t| + |\operatorname{div} u| |\nabla d_t|^2 dx \\
&\leq C \|\nabla u\|_{L^\infty} \|\nabla d_t\|_{L^2}^2; \\
J_{11} &\leq C \|u_t\|_{L^6} \|\nabla d_t\|_{L^2} \|\Delta d\|_{L^3} + C \|u_t\|_{L^2} \|\nabla d_t\|_{L^2} (\|d\|_{L^\infty}^2 + 1) \|d\|_{L^\infty} \\
&\leq C \|\nabla u_t\|_{L^2} \|\nabla d_t\|_{L^2} \|d\|_{H^2}^{\frac{1}{2}} \|\nabla d\|_{H^2}^{\frac{1}{2}} + C \|\rho^{\frac{1}{2}} u_t\|_{L^2} \|\nabla d_t\|_{L^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1} \|\nabla d_t\|_{L^2}^2 (\|\nabla d\|_{H^2}^2 + 1) + \varepsilon \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + C\varepsilon^{-1} \|\nabla d_t\|_{L^2}^2 \\
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + \varepsilon \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + C\varepsilon^{-1} \|\nabla d_t\|_{L^2}^2 (\|\nabla d\|_{H^2}^2 + 1);
\end{aligned}$$

and

$$J_{13} \leq C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1}.$$

To estimate the terms  $J_3, J_4, J_5$  and  $J_{12}$ , by using the standard elliptic estimate on (2.16) and making use of the liquid crystal equation (1.1)<sub>3</sub> yield that

$$\begin{aligned}
\|u\|_{H^2} &\leq C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|\nabla P\|_{L^2} + \|(\nabla d)^T(\Delta d - f(d))\|_{L^2}) \\
&\leq C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|\nabla P\|_{L^2} + \|(\nabla d)^T(d_t + (u \cdot \nabla)d)\|_{L^2}) \\
&\leq C(\|\rho^{\frac{1}{2}} u_t\|_{L^2} + \|\rho\|_{L^6} \|u\|_{L^6} \|\nabla u\|_{L^6} + \|\nabla \rho\|_{L^2} + \|(\nabla d)^T(d_t + (u \cdot \nabla)d)\|_{L^2}) \\
&\leq C(\|\rho^{\frac{1}{2}} u_t\|_{L^2} + \sigma \|\nabla u\|_{H^1} + \sigma^{-1} \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla d\|_{L^3} \|d_t\|_{L^6} + \|\nabla d\|_{L^2}^2 \|u\|_{L^6}) \\
&\leq C(\|\rho^{\frac{1}{2}} u_t\|_{L^2} + \sigma \|\nabla u\|_{H^1} + \sigma^{-1} \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla d_t\|_{L^2}),
\end{aligned}$$

where we have used the estimates (2.6) and (2.10) in the last inequality. Taking  $\sigma$  small enough yields

$$\|u\|_{H^2} \leq C(\|\rho^{\frac{1}{2}} u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla d_t\|_{L^2}). \quad (2.43)$$

Making use of estimates (2.14) and (2.43), we can estimate  $J_3, J_4, J_5$  and  $J_{12}$  as

$$\begin{aligned}
J_3 + J_4 + J_5 &= \int_{\Omega} \rho |u| |\nabla u|^2 |u_t| + \rho |u|^2 |\nabla^2 u| |u_t| + \rho |u|^2 |\nabla u| |\nabla u_t| dx \\
&\leq C(\|\rho\|_{L^\infty} \|u\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} + \|\rho\|_{L^\infty} \|u_t\|_{L^6} \|u\|_{L^6}^2 \|\nabla^2 u\|_{L^2} \\
&\quad + \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2}) \\
&\leq C \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2}^2 \|u\|_{H^2} \leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1} \|u\|_{H^2}^2 \\
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1} (\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) \\
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1} (\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + 1); \\
J_{12} &\leq C \|\nabla \rho\|_{L^2} \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \leq C \|\nabla u_t\|_{L^2} \|u\|_{H^2} \\
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1} \|u\|_{H^2}^2 \\
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1} (\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) \\
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1} (\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + 1)
\end{aligned}$$

Substituting all the estimates of  $J_j$  into (2.42), and taking  $\varepsilon$  small enough, we obtain

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega} (\rho |u_t|^2 + |\nabla d_t|^2) dx + \|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2 \\
&\leq C(\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + C) + \|\nabla d_t\|_{L^2}^2 (\|\nabla u\|_{L^\infty} + \|\nabla d\|_{H^2}^2 + C) + \|\nabla d_t\|_{L^2}^2 + 1) \\
&\leq C(\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) (\|\nabla u\|_{L^\infty} + \|\nabla d\|_{H^2}^2 + C) + C(\|\nabla d_t\|_{L^2}^2 + 1). \quad (2.44)
\end{aligned}$$

Applying the Gronwall's inequality to estimate (2.44), we deduce

$$\sup_{0 \leq t \leq T} \int_{\Omega} (\rho |u_t|^2 + |\nabla d_t|^2) dx + \int_0^T \|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2 dt$$

$$\leq C \int_0^T (\|\nabla d_t\|_{L^2}^2 + 1) dt \exp\left\{\int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla d\|_{H^2}^2 + C) dt\right\} \leq C, \quad (2.45)$$

where we have used estimate (2.15) and the assumption (2.1) in the last inequality. This completes the proof of Lemma 2.5.  $\square$

The following lemma gives the higher order norm estimates of  $u$ ,  $d$  and  $\rho$ .

**Lemma 2.6** *Under the assumption (2.1), it holds that for  $0 \leq T < T^*$*

$$\sup_{0 \leq t \leq T} (\|u\|_{H^2} + \|\nabla d\|_{H^2}) \leq C; \quad (2.46)$$

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6} + \int_0^T \|\nabla^2 u\|_{L^6}^2 dt \leq C. \quad (2.47)$$

**Proof.** From estimates (2.14), (2.38) and (2.43), we have

$$\|u\|_{H^2} \leq C(\|\rho^{\frac{1}{2}} u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla d_t\|_{L^2}) \leq C. \quad (2.48)$$

By using estimates (2.31) and (2.33), we have

$$\begin{aligned} \|\nabla d\|_{H^2} &\leq C(\|\nabla^3 d\|_{L^2} + \|\nabla d\|_{L^2}) \\ &\leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \|u\|_{L^2} \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^2}) \\ &\leq C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^2} + C) \leq C, \end{aligned} \quad (2.49)$$

where we have used the estimates (2.10), (2.14) and (2.38) in the last inequality. Combining the estimates (2.48) and (2.49) above gives the estimate (2.46).

Applying the operator  $\nabla$  to the mass conservation equation (1.1)<sub>1</sub>, then multiplying the resulting equation by  $6|\nabla \rho|^4 \nabla \rho$  and integrating over  $\Omega$  give

$$\begin{aligned} \frac{d}{dt} \|\nabla \rho\|_{L^6}^6 &= -6 \int_{\Omega} |\nabla \rho|^6 \nabla u dx - \int_{\Omega} \nabla(|\nabla \rho|^6) \cdot u dx - 6 \int_{\Omega} |\nabla \rho|^6 \operatorname{div} u dx - 6 \int_{\Omega} \rho |\nabla \rho|^4 \nabla \rho \nabla \operatorname{div} u dx \\ &\leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^6}^6 + C \|\nabla \operatorname{div} u\|_{L^6} \|\nabla \rho\|_{L^6}^5, \end{aligned}$$

that is

$$\frac{d}{dt} \|\nabla \rho\|_{L^6} \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^6} + C \|\nabla \operatorname{div} u\|_{L^6}. \quad (2.50)$$

By using the Gronwall's inequality to the above estimate gives

$$\begin{aligned} \|\nabla \rho\|_{L^6} &\leq (\|\rho_0\|_{W^{1,6}} + C \int_0^T \|\nabla \operatorname{div} u\|_{L^6} dt) \exp\left\{C \int_0^T \|\nabla u\|_{L^\infty} dt\right\} \\ &\leq C \left(\int_0^T \|\nabla^2 u\|_{L^6} dt + 1\right). \end{aligned} \quad (2.51)$$

Applying the standard elliptic regularity result  $\|\nabla^2 u\|_{L^6} \leq C \|\Delta u\|_{L^6}$ , Hölder inequality, Sobolev embedding, the estimates (2.10) and (2.46), we have

$$\|\nabla^2 u\|_{L^6} \leq C(\|\rho u_t\|_{L^6} + \|\rho u \cdot \nabla u\|_{L^6} + \|\nabla P\|_{L^6} + \|(\nabla d)^T (\Delta d - f(d))\|_{L^6})$$

$$\begin{aligned}
&\leq C(\|\nabla u_t\|_{L^2} + \|u\|_{L^\infty}\|\nabla u\|_{L^6} + \|\nabla \rho\|_{L^6} + \|\nabla d\|_{L^\infty}\|\Delta d\|_{L^2} + \|\nabla d\|_{L^6}\|f(d)\|_{L^\infty}) \\
&\leq C(\|\nabla u_t\|_{L^2} + \|u\|_{H^2}^2 + \|\nabla \rho\|_{L^6} + \|d\|_{H^3}^2 + \|d\|_{H^2}) \\
&\leq C(\|\nabla u_t\|_{L^2} + \|\nabla \rho\|_{L^6} + 1).
\end{aligned} \tag{2.52}$$

Inserting the estimate (2.52) into (2.51) yields

$$\begin{aligned}
\|\nabla \rho\|_{L^6} &\leq C \int_0^T (\|\nabla u_t\|_{L^2} + \|\nabla \rho\|_{L^6} + 1) dt \\
&\leq C \int_0^T (\|\nabla u_t\|_{L^2}^2 + \|\nabla \rho\|_{L^6} + 1) dt \leq C \int_0^T \|\nabla \rho\|_{L^6} dt + C,
\end{aligned}$$

where we have used the estimate (2.38), then applying the Gronwall's inequality gives

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6} \leq C. \tag{2.53}$$

From (2.52) and (2.53), we have

$$\int_0^T \|\nabla^2 u\|_{L^6}^2 dt \leq C \left( \int_0^T \|\nabla u_t\|_{L^2}^2 dt + \sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6}^2 + C \right) \leq C. \tag{2.54}$$

It is easy to know that the estimate (2.48) follows (2.53) and (2.54) immediately. This completes the proof of Lemma 2.6.  $\square$

We now give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** From the existence result of Theorem 1.1, we know that  $\|u(t)\|_{H^2}$ ,  $\|\rho(t)\|_{W^{1,6}}$ ,  $\|d(t)\|_{H^3}$  and  $\|\rho^{\frac{1}{2}}u_t(t)\|_{L^2}$  are all continuous on the time interval  $[0, T^*)$ . From the above Lemmas 2.1–2.6, we see that for all  $T \in (0, T^*)$ ,

$$(\|u\|_{H^2}, \|\rho\|_{W^{1,6}}, \|d\|_{H^3}, \|\rho^{\frac{1}{2}}u_t\|_{L^2})(T) \leq C. \tag{2.55}$$

Furthermore, there hold

$$\rho^{\frac{1}{2}}u_t + \rho^{\frac{1}{2}}u \cdot \nabla u \in L^\infty([0, T^*]; L^2), \tag{2.56}$$

and for all  $T \in (0, T^*)$ ,

$$(\mu \Delta u - \lambda \operatorname{div}(\nabla d \odot \nabla d - \frac{1}{2}(|\nabla d|^2 + F(d))) - \nabla P)(T) = (\rho u_t + \rho u \cdot \nabla u)(T) = \sqrt{\rho}g, \tag{2.57}$$

where  $g(T) \triangleq (\rho^{\frac{1}{2}}u_t + \rho^{\frac{1}{2}}u \cdot \nabla u)(\cdot, T) \in L^2$ . Therefore, from (2.56) and (2.57), we can take  $(\rho, u, d)|_{t=T}$  with any  $T \in (0, T^*)$  as the initial data and apply Theorem 1.1 to extend the local strong solution to a time interval  $[T, T + \delta]$  for a uniform  $\delta > 0$  which only depends on the bounds obtained in these lemmas, so that the solution can be extended to the time interval  $[0, T^* + \delta]$ . This contradicts with the maximality of  $T^*$ . Hence, the assumption (2.1) cannot be true. This completes the proof of Theorem 1.2.  $\square$



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